Intergrerend project dynamische systemen

4 April 2014

university of groningen

Exam Problem Sheet

The exam consists of 4 problems. You may answer in Dutch or in English. You have 90 minutes to answer the questions. Give brief but precise answers. You can achieve 50 points in total which includes a bonus of 5 points.

1. **[3+5+2 Points.]**

   Consider the one-dimensional growth model

   $$x' = ax - h\sin t,$$

   where a and h are positive real constants.

   (a) Determine the general solution and show that the system has exactly one periodic solution.

   (b) Determine the Poincaré map of the system and show that it has exactly one fixed point which is always a source.

   (c) Use the results in part (b) to sketch the solution curves of the original time continuous system in the (t, x)-plane.

2. **[3+3+4+2 Points.]**

   Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. Then the system

   $$X' = -\nabla V(X) \tag{1}$$

   is called a *gradient system* (here $\nabla V(X) = (\frac{\partial}{\partial x_1}V(X), \dots, \frac{\partial}{\partial x_n}V(X))$ with $X = (x_1, \dots, x_n) \in \mathbb{R}^n$). It is clear that the equilibrium points of a gradient system are given by the critical points of V.

   (a) Show that if X is not an equilibrium point, then V is strictly decreasing along the solution curve through X.

   (b) State the definition of asymptotic stability for the equilibrium point of a time continuous system.

   (c) Show that if $X^\star$ is an isolated minimum of V then $X^\star$ is asymptotically stable. What can you say about the basin of attraction of $X^\star$ in this case?

   (d) What are the conditions on V at an equilibrium point $X^\star$ to conclude stability from the linearization of the gradient system at $X^\star$?

— please turn over —

3. **[2+4+5 Points.]**

Consider the planar system

$$X' = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} X\,.$$

(a) Determine the canonical form of the system.

(b) Determine the phase portraits of the original system and its canonical form.

(c) Determine a conjugacy relating the flows of the system to its canonical form.

4. **[3+9 Points.]**

(a) State the definition of chaos for a discrete time system.

(b) Argue that the doubling map

$$d : [0,1] \to [0,1], \quad x \mapsto 2x \bmod 1$$

is chaotic.